Math 126 Midterm 2 Practice

UCB Summer 2026

July 22, 2026

Remember that the exam is closed book. You will have 50 minutes to complete the Exam. You may use the front and back of each page. There are 3 total problems, so ensure that you have attempted all 3. Please write your full name and SID below. Good luck!

Name:

SID:

Honor Pledge:

Formulas: Characteristic ODEs:

$$\begin{aligned} x'(s) &= D_p F \\ z'(s) &= D_p F \cdot p \\ p'(s) &= -D_x F - D_z(F) p \end{aligned}$$

Divergence:

$$\int_\Omega \nabla \cdot F dx = \int_{\partial\Omega} F \cdot \eta dS$$

D'Alembert:

$$u(t,x) = \frac{1}{2}\left[g(x+ct) + g(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} h(\tau) d\tau$$

$$\mathcal{D}_{t,x} = \{(s,y) \in \mathbb{R}_+ \times \mathbb{R} \mid x - c(t-s) \leq y \leq x + c(t-s)\}$$

Inhomogeneous Wave:

$$u(t,x) = \frac{1}{2c}\int_{\mathcal{D}_{t,x}} f(s,y) ds dy$$

Poisson:

$$u(t,x) = \frac{\partial}{\partial t}\left(\frac{t}{2\pi}\int_D \frac{g(t-xy)}{\sqrt{1-|y|^2}} dy\right) + \frac{t}{2\pi}\int_D \frac{h(t-xy)}{\sqrt{1-|y|^2}} dy$$

Kirchoff:

$$u(t,x) = \frac{\partial}{\partial t}\left(\frac{1}{4\pi t}\int_{\partial B(x,t)} g(y) dS(y)\right) + \frac{1}{4\pi t}\int_{\partial B(x,t)} h(y) dS(y)$$

Helmholtz 1D:

$$\phi_k(x) = \sin(\frac{k\pi}{l} x), \qquad k \in \mathbb{Z}$$

solve

$$-\phi(x) = \lambda\phi \qquad \phi(0) = \phi(l) = 0$$

for $\lambda = (\frac{k\pi}{l})^2$.

Helmholtz Solutions on the Torus:

$$\phi_k(x) = e^{ikx} \qquad k \in \mathbb{Z}$$

Heat Separation of Variables on the Torus: $u(t,x) = e^{-k^2 t} e^{ikx}$ for $k \in \mathbb{Z}$.

Heat Kernel:

$$H_t(x) = (4\pi t)^{-n/2} e^{-\frac{|x|^2}{4t}}$$

Heat Solution on $\mathbb{R}^n$ with initial condition $f \in C^0(\mathbb{R}^n)$:

$$u(t,x) = H_t(x) * f(x)$$

Fourier Coefficient:

$$c_k[f] = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-ikx} dx$$

Poisson Kernel:

$$P_r(\theta) = \sum_{\mathbb{Z}} r^{|k|} e^{ik\theta}$$

Laplace Solution with boundary condition g:

$$u(r,\theta) = P_r(\theta) * g(\theta)$$

Problem 1:

True/False
Part A.) $C_c^\infty(\Omega)$ is a complete subspace of $L^1(\Omega)$ (in the L^1 norm).
Part B.) Under the mixed boundary conditions $u(0) = 0$, $u'(l) = 0$ on $(0, l)$, the Laplacian is self-adjoint.
Part C.) The partial Fourier sums of a function $f \in C^1(\mathbb{T})$ converge pointwise to f.
Part D.) If $\sum |c_k[f]^2 k^4| < \infty$, then f must be C^2.
Part E.) The heat equation may be used to describe the motion of particles.

Problem 2:

Let $u(t, x)$ be the solution to the heat equation on $(0, \infty) \times \mathbb{R}$ given by

$$u(t, x) = H_t(x) * g(x)$$

for $g \in C_c^\infty(\mathbb{R})$ so $\text{supp}(g) \subset [-1, 1]$ and $0 \leq g(x) \leq 1$. Show that

$$\lim_{t \to \infty} u(t, x) = 0$$

for all x.

Extension: what bounds can we put on $u(t, x)$ compared to g?

Problem 3:

Suppose $u, \phi \in C^2(\Omega) \cap C^0(\overline{\Omega})$ for a bounded domain $\Omega \subset \mathbb{R}^n$. Let u be subharmonic and ϕ harmonic, so $u|_{\partial\Omega} \leq \phi|_{\partial\Omega}$. Show $u \leq \phi$ on Ω.

Problem 4:

Let $g(\theta) = (\theta - \pi)^2$ on $[0, 2\pi)$. Does the Laplace equation admit a solution on $B(0, 1) \subset \mathbb{R}^2$ with this boundary data? If so, what is it (your solution will be a sum).

Problem 5:

Let $u(t, x)$ be a heat solution on the torus with initial condition f, so

$$u(t, x) = \sum_{\mathbb{Z}} c_k[f] e^{-k^2 t} e^{ikx}$$

Under what condition on f does $u(t, x)$ decay in time?

Problem 6:

Recall that $\{\frac{1}{\sqrt{2\pi}} e^{ikx}\}$ is a basis for $L^2(-\pi, \pi))$. Show that the functions $\phi_k(x) = \sqrt{\frac{2}{\pi}} \sin(kx)$ form a basis for $L^2((0, \pi))$, assuming the ϕ_k already have magnitude 1. *Hint:* Any function on $L^2((0, \pi))$ may be extended to an odd function on $(-\pi, \pi)$.